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Stochastic Reachability as an Exit Problem

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Abstract— **For stochastic hybrid systems, safety verification methods are very little supported mainly because of complexity and difficulty of the associated mathematical problems. The key of the methods that succeeded in solving various instances of this problem is to prove the equivalence of these instances with known problems. In this paper, we apply the same pattern to the most general model of stochastic hybrid systems. Stochastic reachability problem can be treated as an exit problem for a suitable class of Markov processes. The solutions of this problem can be characterised using Hamilton Jacobi theory.**

I. Introduction

Stochastic hybrid systems are a class of non-linear stochastic continuous time/space hybrid dynamical systems. For these systems different models have been developed by many researchers in the field of hybrid systems. These models can be used to analyse and design complex embedded systems that operate in the presence of variability and uncertainty, and incorporate complex (hybrid/stochastic) dynamics, randomness, multiple modes of operations. Under some natural assumptions on their parameters, their behaviour can be described by stochastic processes having good properties. A very important verification problem for such systems consists mainly in reachability analysis. The aim of reachability analysis is to determine the probability that the system will reach a set of desirable/unsafe states, and the difficulty of this problem comes from the interaction between discrete/continuous dynamics and the active boundaries.

In the literature, for deterministic hybrid systems there exist different methods to deal with the reachability problem. The most used methods are based on optimal control (Hamilton Jacobi equations) such that the computational issues are solved using dynamic programming. As well, reachability problem for hybrid systems can be thought of as an exit problem from a given domain. This also involves solving a standard Hamilton-Jacobi-Bellman equation over this set and possibly pieces of its boundary with rather complicated boundary conditions (see the discussions from [21] and the references therein).

In the stochastic hybrid system framework, it has been proved that aiming to tackle stochastic reachability as an optimal control problem could be a very challenging and difficult task [1]. The main explanation for this difficulty can be found in the structure of the stochastic processes that describe the behaviour of stochastic hybrid systems. These processes are Markovian processes with piecewise continuous paths. Their discontinuities are describe by some spontaneous jumps (in a Poisson style) and forced jumps dictated by some guards. In mathematics these forced jumps are called predictable jumps. Their presence leads to some discontinuities of the transition probabilities of the Markov processes considered in this context. The main problem comes from the fact that the dynamic programming theory for the Markov processes (that describe the behaviour of SHS) with predictable jumps is not fully understood and developed. In most cases, dynamic programming methods are applied locally to these processes where they behave nicely like some diffusion processes or, more general, Feller-Markov processes [17].

For SHS, the stochastic reachability problem means to compute the probability of the set of those traces that start with a given probability distribution and hit in a finite/infinite horizon time a target set. In many papers [19], [23], [18], the standard methodology to approach this problem is to approximate the stochastic process that corresponds to the given hybrid system by simpler processes (like Markov chains) and then to derive convergence results for the reach set probabilities. Also, from a computer science perspective, Markov chain approximations are desirable for probabilistic model checking. Due to the complexity of such models, the Markov chain approximations suffer from state space explosion (see [17] and discussions therein). Then, at this point we are wondering what kind of approximations suit better to stochastic hybrid processes. Stochastic hybrid processes are jump type Markov processes. From the control theory and stochastic analysis perspectives, it seems that such processes are better studied using diffusion approximations [15].

In this paper, we characterize stochastic reachability either using expectations of the hitting times for the target sets, either using probability distribution functions as solutions of the forward Kolmogorov equation associated with the underlying Markov process. In both cases, the main idea is that the reachability problem can be treated as an exit problem. In the first case, it can be proved that the reach set probability is exactly the expectation of the probabilistic event that the first hitting time of the target set (or the exit time from the complement of this set) is less than the horizon time. In the second case, we look at the probability distribution functions of those sets that cover the complement of the target set. In both cases, the quantities involved can be characterised as solutions for some appropriate Hamilton

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Jacobi equations [13].

In the mathematical literature, the problem of computing the escape rate and of the probability distribution of the escape points on the boundary of a given domain is referred to as Kolmogorov's exit problem. There exists a very rich literature [11], [12], [22], [25] regarding the asymptotic estimations for exit time probabilities associated with different classes of Markov processes. This provides us techniques, ideas and methodologies that can help in dealing with the exit problems for SHS. Moreover, many of these classes of Markov processes can be considered particular stochastic hybrid processes (Markov processes with Levy generators, dynamical systems driven by some Markov jump processes, jump-diffusions).

II. PRELIMINARIES

A. Markov Processes

Let us consider $M = (x_t, P_x)$ a Markov process with the state space X. A Markov process retains no memory of where it has been in the past. Standard definitions can be find in any textbook [9], [10]. Let $\mathcal{F}$ and $\mathcal{F}_t$ be the appropriate completion of σ-algebras $\mathcal{F}^0 = \sigma\{x_t | t \geq 0\}$ and $\mathcal{F}^0_t = \sigma\{x_s | s \leq t\}$. $\mathcal{F}_t$ describes the history of the process up to the time t. Technically, with any state $x \in X$ we can associate a natural probability space $(\Omega, \mathcal{F}, P_x)$ where P_x is such that its initial probability distribution is $P_x(x_0 = x) = 1$. Strong Markov property means that the Markov property is still true with respect to the stopping times of the process M.[1] In particular, any Markov chain is a strong Markov process.

We adjoin an extra point Δ (the cemetery) to X as an isolated point, $X_\Delta = X \cup \{\Delta\}$. The existence of Δ is assumed in order to have a probabilistic interpretation of $P_x(x_t \in X) < 1$, i.e. at some 'termination time' $\zeta(\omega)$ when the process M escapes to and is trapped at Δ.

X is equipped with Borel σ-algebra $\mathcal{B}(X)$ or shortly $\mathcal{B}$. Consider the set $\mathbf{B}(X)$ of bounded real measurable functions defined on X, which is a Banach space with the sup-norm $||\varphi|| = \sup_{x\in X} |\varphi(x)|$, $\varphi \in \mathbf{B}(X)$. The semigroup of operators (P_t) is given by

$$P_t f(x) = E_x f(x_t) = \int f(y) p_t(x, dy), t \geq 0 \qquad (1)$$

where E_x is the expectation with respect to P_x and $p_t(x, A)$, $x \in X$, $A \in \mathcal{B}$ represent the transition probabilities, i.e. $p_t(x, A) = P_x(x_t \in A)$. The semigroup property of (P_t) can be derived from the Chapman-Kolmogorov equations satisfied by the transition probabilities. The infinitesimal generator of , denoted by L, is the derivative of P_t at $t = 0$. Let $D(L) \subset \mathcal{B}^b(X)$ be the set of functions f for which the following limit exists (denoted by Lf)

$$\lim_{t \searrow 0} \frac{1}{t}(P_t f - f) \qquad (2)$$

In most cases, the operator semigroup can be itself characterized by its infinitesimal generator. When $D(L)$ is large enough, the infinitesimal generator captures the law of the whole dynamics of a Markov process and provides a tool to study the Markov process.

[1] Recall that a $[0, \infty]$-valued function τ on Ω is called an $\{\mathcal{F}_t\}$-*stopping time* if $\{\tau \leq t\} \in \mathcal{F}_t, \forall t \geq 0$.

B. Kolmogorov Backward and Forward Equations

This subsection recalls some basic facts concerning the backward and forward Kolmogorov equation for Markov processes. The forward equation is also known as the Fokker Planck Kolmogorov equation for diffusion processes. The Fokker Planck equation is one of the basic tools when dealing with diffusion processes, because it allows to calculate the probability density function (pdf) ρ_t of the process at time $t \geq 0$ given an initial probability density ρ_0 and eventually the stationary pdfs (when they exist).

The semigroup (P_t) of a Markov process M satisfies the following differential equation: for all $f \in D(L)$,

$$\frac{d}{dt} P_t f = L P_t f. \qquad (3)$$

This equation is called *Kolmogorov's backward equation* [9]. In particular, if we define the function $u(t, x) = P_t f(x)$ then u is solution of the PDE

$$\begin{cases} \frac{\partial u}{\partial t} = Lu \\ u(0, x) = f(x). \end{cases}$$

Conversely, if this PDE admits a unique solution, then its solution is given by $P_t f(x)$. Moreover, it is easy to check that the operators P_t and L commute. Then (3) may be written as

$$\frac{d}{dt} P_t f = P_t L f. \qquad (4)$$

This equation is known as *Kolmogorov's forward equation*. It is the weak formulation of the equation $\frac{d}{dt}\mu^x_t = L^* \mu^x_t$, where the probability measure μ^x_t on X denotes the law of (x_t) conditioned on $x_0 = x$ and where L^* is the adjoint operator of L.

In particular, if M is a diffusion process on $\mathbb{R}^n$ and if $\mu^x_t(dy)$ admits a density $q(x; t, y)$ w.r.t. the Lebesgue measure, the forward Kolmogorov equation is the weak form (in the sense of distribution theory) of the PDE

$$\begin{aligned} \frac{\partial}{\partial t} q(x; t, y) &= -\sum_{i=1}^{n} \frac{\partial}{\partial y_i}(b_i(y) q(x; t, y)) + \qquad (5) \\ &\quad + \sum_{i,j=1}^{d} \frac{\partial^2}{\partial y_i \partial y_j}(a_{ij}(y) q(x; t, y)) \end{aligned}$$

where $b_i(x)$ and $a_{ij}(x)$ are respectively the drift coefficient and the diffusion coefficient of the process. The equation (5) is known as the Fokker-Planck equation associated to a diffusion process.

III. STOCHASTIC HYBRID SYSTEMS

We adopt the General Stochastic Hybrid System model presented in [6], [5]. This subsection describes the model and establishes the notation.

Let Q be a set of discrete states. For each $q \in Q$, we consider the Euclidean space $\mathbb{R}^{d(q)}$ with dimension $d(q)$ and we define an *invariant* as an open subset X^q of $\mathbb{R}^{d(q)}$. The hybrid state space is the set $X(Q,d,\mathcal{X}) = \bigcup_{i\in Q}\{i\} \times X^i$ and $x = (i, z^i) \in X(Q,d,\mathcal{X})$ is the hybrid state. The closure of the hybrid state space will be $\overline{X} = X \cup \partial X$, where $\partial X = \bigcup_{i\in Q}\{i\} \times \partial X^i$. It is known that X can be endowed with a metric ρ whose restriction to any component X^i is equivalent to the usual component metric [9]. Then $(X, \mathcal{B}(X))$ is a Borel space (homeomorphic to a Borel subset of a complete separable metric space), where $\mathcal{B}(X)$ is the Borel σ-algebra of X. Let $\mathbf{B}(X)$ be the Banach space of bounded positive measurable functions on X with the norm given by the supremum.

A (General) Stochastic Hybrid System (SHS) is a collection $H = ((Q,d,\mathcal{X}),(b,\sigma),\mu_0,(\lambda,R))$, where

• $(Q,d,\mathcal{X})$ describes the hybrid state space: Q is a countable/finite set of discrete states (modes); $d : Q \to \mathbb{N}$ is a map giving the dimensions of the continuous state spaces; $\mathcal{X} : Q \to \mathbb{R}^{d(\cdot)}$ maps each $q \in Q$ into an open subset X^q of $\mathbb{R}^{d(q)}$;

• (b,σ) provides the coefficients of the diffusion part (continuous dynamics in modes): $b : X(Q,d,\mathcal{X}) \to \mathbb{R}^{d(\cdot)}$ is a vector field; $\sigma : X(Q,d,\mathcal{X}) \to \mathbb{R}^{d(\cdot)\times m}$ is a $X^{(\cdot)}$-valued matrix, $m \in \mathbb{N}$,

• μ_0 is the initial probability measure defined on $(X, \mathcal{B}(X))$;

• (λ, R) gives the jumping mechanism: $\lambda : \overline{X}(Q,d,\mathcal{X}) \to \mathbb{R}^+$ is a transition rate function; $R : \overline{X} \times \mathcal{B}(\overline{X}) \to [0,1]$ is a stochastic kernel that provides the post-jump location.

The realization of an SHS is built as a *Markov string* [6] obtained by the concatenation of the paths of some diffusion processes (z_t^i), $i \in Q$ together with a jumping mechanism given by a family of stopping times (S^i). Let ω_i be a diffusion trajectory, which starts in $(i, z^i) \in X$. Let $t_*(\omega_i)$ be the first hitting time of ∂X^i of the process (x_t^i). Define the function

$$F(t,\omega_i) = I_{(t<t_*(\omega_i))} \exp(-\int_0^t \lambda(i, z_s^i(\omega_i))ds).$$

This function will be the *survivor function* for the stopping time S^i associated to the diffusions (z_t^i).

A stochastic process $x_t = (q(t), z(t))$ is called an *SHS realization* if there exists a sequence of stopping times $T_0 = 0 < T_1 < T_2 \leq \ldots$ such that for each $k \in \mathbb{N}$,

• $x_0 = (q_0, z_0^{q_0})$ is a $Q \times X$-valued random variable chosen according to the probability distribution μ_0;

• For $t \in [T_k, T_{k+1})$, $q_t = q_{T_k}$ is constant and $z(t)$ is a solution of the stochastic differential equation (SDE):

$$dz(t) = b(q_{T_k}, z(t))dt + \sigma(q_{T_k}, z(t))dW_t \tag{6}$$

where W_t is a the m-dimensional standard Wiener process;

• $T_{k+1} = T_k + S^{i_k}$ where S^{i_k} is chosen according to the survivor function F.

• The post jump location $x(T_{k+1})$ is sampled according to the probability distribution $R\left((q_{T_k}, z(T_{k+1}^-)), \cdot\right)$.

The realization of any SHS, H, under standard assumptions (about the diffusion coefficients, non-Zeno executions, transition measure, etc, see [6] for a detailed presentation) is a strong Markov process. Let $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, P_x)$ be the strong Markov process associated to H. The sample paths of M are right continuous with left limit, i.e. cadlags.

The infinitesimal generator of an SHS is an integro-differential operator. It has been proved in [5] that the extended generator of an SHS has the following expression:

$$\mathcal{L}f(x) = \mathcal{L}_{cont}f(x) + \lambda(x)\int_{\overline{X}}(f(y) - f(x))R(x,dy) \tag{7}$$

where $\mathcal{L}_{cont}f(x)$ has the standard form of the diffusion infinitesimal operator. What makes this generator different from the generator of a Feller Markov process (like a diffusion process) is its domain that contains at least the set of second order differentiable functions that satisfy the boundary condition, as follows:

$$f(x) = \int_{\mathbb{X}} f(y)R(x,dy),\ x \in \partial X.$$

In the presence of forced jumps, the generator of an SHS is an operator that is difficult to deal with, since its domain does not even contain the set of all compactly supported C^∞ functions.

IV. STOCHASTIC REACHABILITY

Let us consider $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, P_x)$ being a (strong right) Markov process, the realization of a stochastic hybrid system H. For this strong Markov process we address a verification problem consisting of the following *stochastic reachability problem.*

Given a target set, the objective of the reachability problem is to compute the probability that the system trajectories from an arbitrary initial state will reach the target set.
Formally, given a set $A \in \mathcal{B}(X)$ and a time horizon $T > 0$, let us define:

$Reach_T(A) = \{\omega \in \Omega \mid \exists t \in [0,T] : x_t(\omega) \in A\}$
$Reach_\infty(A) = \{\omega \in \Omega \mid \exists t \geq 0 : x_t(\omega) \in A\}$.

These two sets are the sets of trajectories of M, which reach the set A (the flow that enters A) in the interval of time $[0,T]$ or $[0,\infty)$. The reachability problem consists of determining the probabilities of such sets. The probabilities of reach events are

$$P(T_A < T) \text{ or } P(T_A < \zeta) \tag{8}$$

where ζ is the life time of M and T_A is the first hitting time of A

$$T_A = \inf\{t > 0 | x_t \in A\} \tag{9}$$

and P is a probability on the measurable space $(\Omega, \mathcal{F})$ of the elementary events associated to M. P can be chosen to be P_x (if we want to consider the trajectories that start in x) or P_{μ_0} ((if we want to consider the trajectories with an initial condition chosen according to an initial probability distribution μ_0).

Denote by P_A the *hitting operator* associated to the underlying Markov process (x_t), i.e.

$$P_A v = E_x\{v \circ x_{T_A} | T_A < \zeta\} \tag{10}$$

and T_A is given by (9).

Proposition 1: [7] For any $x \in X$ and Borel set $A \in \mathcal{B}(X)$, we have

$$P_x[Reach_\infty(A)] = P_A 1(x) = P_x[T_A < \zeta].$$

Note that the first hitting time of A is equal with the first exit time from the complementary set of A, $E = A^c = X \backslash A$. Then, the stochastic reachability problem can be formulated as an exit problem for the right Markov processes that appear as realizations of SHS. These processes may be viewed as piecewise continuous jump diffusions, where the jumps are allowed to be spontaneous, or forced (predictable). For continuous pure diffusions processes, it is sufficient to consider the time when the process hits the boundary of E or A. However, when the stochastic processes also includes jumps, then it is possible that the process overshoots the boundary and ends up in the exterior of the domain E (i.e. in the interior of A).

An important quantity that can be computed without explicitly constructing the transition probability density function is the mean first passage time of the process from a specified domain. The mean first passage time is a measure of the stochastic time scale for the process to be in a specified domain. The mean first passage time is a solution of a boundary value problem involving the backward Kolmogorov operator (the adjoint of the operator in the forward equation). If the process is a continuous diffusion with the infinitesimal generator L and the target set A is a closed set, it is known that if the PDE (Dirichlet problem)

$$\begin{cases} \frac{\partial u}{\partial t} = Lu & \text{on } E \times (0,T] \\ u = 0 & \text{on } E \times \{0\} \\ u = \mathbf{1}_{\partial E} & \text{on } \partial E \times (0,T] \end{cases}$$

has a bounded solution, then

$$u(x,t) = P_x\{T_A \leq t, x_{T_A} \in \partial E\}, 0 \leq t \leq T. \quad (11)$$

A possible approach is to consider another process $\widetilde{x}_t$ that coincides with x_t up to time T_A^- and then goes to a terminal state Δ. Then

$$P_{\mu_0}(T_A < T) = P_{\mu_0}(\widetilde{x}_T = \Delta) = q(\Delta, T)$$

and $q(\Delta, T)$ can be computed as solution for the Fokker Planck Kolmogorov equation associated to the diffusion process.

Let us consider the logarithmic transformation of reachability function $u(x,t)$ given by (11), i.e.

$$h = -\ln u$$

This implies that u is the Laplace transform of h, i.e. $u = e^{-h}$. Then h is the solution of the following Hamilton Jacobi equation [13]:

$$\begin{cases} -\frac{\partial h}{\partial t} - bD_x h + \frac{1}{2} D_x h a D_x h' = 0 & \text{in } E \times (0,T) \\ h(x,t) = 0 & \text{on } \partial E \times [0,T] \\ h(x,t) \to +\infty & \text{as } t \nearrow T \text{ if } x \in E. \end{cases}$$

Let us consider an SHS $H = ((Q, d, \mathcal{X}), (b, \sigma), \mu_0, (\lambda, R))$ defined as in the Section III, with a finite number of modes ($cardQ < \aleph_0$). We can suppose that H has only forced jumps, no spontaneous jumps. This can be achieved introducing an extra variable that "simulates" the spontaneous jumps [9]

Suppose that the target set A is closed and E its complement. Denote $E^q = X^q \cap E$, $q \in Q$. Then $\{\partial E^q | q \in Q\}$ represents a partition of the boundary ∂E. Suppose that $supp\mu_0 \subset E$ (μ_0 is the initial probability measure). The first quantities we need to compute are

$$u^q(t) = P_x(T_A < t, \ x_{T_A} \in \partial E^q\}, \, q \in Q, \, 0 \leq t \leq T.$$

Then, for each $q \in Q$, u^q is solutions of the Dirichlet problem associated to the diffusion process associated to the mode q (see the previous paragraph). The desired ρ^q could be recovered from u^q, if the initial probability distribution for each mode is known.

The purpose of this paper is to study the problem of stochastic reachability as an exit problem for an appropriate stochastic process. For this we need to develop another perspective on reachability as follows.

The exact stochastic reachable set at time T is defined as the probability distribution μ_T of the state (x_t), which is the realization of the SHS H, for an initial probability distribution μ_0. Given a set $A \in \mathcal{B}(X)$ and a time horizon $T > 0$, let us define

- The enclosing hull of all probability distribution μ_t within a time interval $t \in [0,T]$ denoted by $\mu_{[0,T]}$ and given by:

$$\mu_{[0,T]} = \sup\{\mu_t | t \in [0,T], P(x_0 \in \cdot) = \mu_0(\cdot)\}$$

- The over-approximated reachable set probability as

$$\overline{\mu}_{[0,T]}(A) = \sup\{\mu_{[0,T]}(F) | F \in \mathcal{B}(X); F \subseteq E = A^c\}$$

The quantity $\overline{\mu}_{[0,T]}(A)$ can capture the cases when the first hitting time of A is NOT the first hitting time of its boundary. These cases may appear due to the discontinuities of (x_t) when it makes the spontaneous discrete transitions. The main tool in the computation of this type of measure will be the forward Kolmogorov equation associated to our Markovian process.

V. Forward and Backward Kolmogorov Equation for SHS

The realization of an SHS is described by a Markov jump type process. Jump process is understood in a rather large sense, i.e. process with discontinuities in the natural filtration. A complete description of a Markov jump process is given by its transition density function, which is the solution of the forward and backward Kolmogorov equation. The forward Kolmogorov equation is known also as the Fokker Planck equation. Through it is not always possible to find the transition probability density function for a Markov jump process for all times, some important questions can be satisfactorily answered by computing such quantities as the stationary density or the quasi-stationary density. The quasi-stationary density represents an approximation to the eigenfunction corresponding to the smallest eigenvalue of the Kolmogorov operator. Thus it describes the long time behavior of the process.

The backward Kolmogorov equation of a stochastic hybrid process M (realization of an SHS defined as in Section III) is

$$\frac{d}{dt}P_t f = \mathcal{L}_{cont}P_t f(x)+\lambda(x)\int_{\overline{X}}(P_t f(y)-P_t f(x))R(x,dy) \tag{12}$$

A generalised Fokker Planck equation is well known for the case of switching diffusions (where there are no forced transitions). A unifying formulation of the Fokker-Planck-Kolmogorov (FPK) equation for general stochastic hybrid systems is developed in [3]. The FPK equation for SHS is based on the concept of mean jump intensity. Let us define a positive measure J on $X \times (0,\infty)$ by

$$J(A) = E_{\mu_0}\{\sum_{k\geq 0} 1_A(x^-_{T_k}, T_k)\}.$$

For any $\Gamma \in \mathcal{B}$, the quantity $J(\Gamma \times (0,t])$ is the expected number of jumps starting from Γ during the interval $(0,t]$.

Suppose that there exists a mapping $r : t \mapsto r_t$, from $[0,\infty)$ to the set of all bounded measures on X such that for all $\Gamma \in \mathcal{B}$, we have: (a) $t \mapsto r_t(\Gamma)$ is measurable; (b) for all $t \geq 0$,

$$J(\Gamma \times (0,t]) = \int_0^t r_s(\Gamma)ds.$$

Then r is called the mean jump intensity of the process M under the initial law μ_0.

The generalised FPK equation can be written symmetrically as

$$\mu_t^{'} = \mathcal{L}^*_{cont}\mu_t + \int (W_t(dx,\cdot) - W_t(\cdot,dx)) \tag{13}$$

where $W_t(dx,dy) = r_t(dx)R(x,dy)$, or

$$\mu_t^{'} = \mathcal{L}^*_{cont}\mu_t + r_t(R - I)$$

where I is the identity kernel, i.e. $I(x,dy) = \delta_x(dy)$. Here, $\mathcal{L}^*_{cont}$ is the adjoint of $\mathcal{L}_{cont}$ (the continuous part of the infinitesimal operator of M) in the sense of distribution theory.

Remark that in the case of stochastic hybrid processes, the forward and backward Kolmogorov equations are *parabolic integro partial differential equations*.

VI. Dealing with Stochastic Reachability

In this section, we give a theoretical characterization of the reach set probabilities as solutions for some appropriate Hamilton Jacobi equations.

A. Variational inequalities

Let X be a bounded open set in $\mathbb{R}^N$ with *smooth boundary*. $\mathbb{R}^N$ can be thought of as the Euclidean space where the state space of a stochastic hybrid system can be embedded.

According to [2], [20], for the existence of the viscosity solutions some assumptions are necessary. For the *Dirichlet problem* given by (15) and (16), these assumptions can be formulated as follows:

(A.1) $F \in C(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N \times \mathbb{R})$,
(A.2) F satisfies the local and non-local *degenerate ellipticity condition(s)*: for any $x \in \mathbb{R}^N$, $u \in \mathbb{R}$, $p \in \mathbb{R}^N$, $A, B \in \mathcal{S}^N$, $l_1, l_2 \in \mathbb{R}$

$$F(x,u,p,A,l_1) \leq F(x,u,p,B,l_2) \text{ if } A \geq B,\ l_1 \geq l_2$$

(A.3) $R(x,\cdot)$ is a probability measure on X for $x \in \partial X$ such that the linear operator

$$Rv(x) = \int_X v(y)R(x,dy) \tag{14}$$

satisfies

$$|Rv(x)| \leq C||v||_{L^1(X)}, \text{ for all } v \in L^1(X)$$

where C does not depend on v.
(A.4) The function $x \longmapsto Rv(x)$ is continuous w.r.t. $x \in \overline{X}$, uniformly for $v \in L^\infty(X)$.

Motivated by the expression of the generator associated to an SHS, let us consider the linear integro-differential equations of the following form:

$$\partial_t u + F(x,u,D_x u, D_x^2 u, \int_X u(y)R(x,dy)) = 0, \tag{15}$$

where $D_x u$ denotes the space gradient, $D_x^2 u$ the matrix of second derivatives and $R(x,\cdot)$ is a probability kernel. Here, $\mathcal{S}^N$ denotes the space of symmetric $N \times N$ real valued matrices. The applications for (15) are dynamic programming equations associated with the control of the right Markov processes that appear as SHS realizations.

In the case when the state space X is a bounded domain of a Euclidean space, the process jumps back into X upon hitting the boundary, which leads to the following boundary condition to be coupled with the equation (15),

$$u(x) = \int_X u(y)R(x,dy),\ x \in \partial X. \tag{16}$$

For a bounded function $u : \mathbb{R} \times X \to \mathbb{R}$, its upper/lower semicontinuous envelopes can be defined in a standard way [20], [2]. Furthermore, the definitions of the viscosity (sub/super) solutions for second-order parabolic integro-differential equations are well established now in the literature [2].

Let u be a bounded function.
(i) u^* is a *viscosity subsolution* of (15) if

$$\partial_t u^* + F(x,u^*,D_x\phi, D_x^2\phi, \int_X u^*(y)R(x,dy) \leq 0$$

for any $\phi \in C^2(X)$ and any local maximum x for $u^* - \phi$.
(ii) u_* is a *viscosity supersolution* of (15) if

$$\partial_t u_* + F(x,u_*,D_x\phi, D_x^2\phi, \int_X u_*(y)R(x,dy) \geq 0$$

for any $\phi \in C^2(X)$ and any local minimum x for $u_* - \phi$.
(iii) u is a *viscosity solution* if u is a viscosity sub- and supersolution.

A bounded function $u : \mathbb{R}\times\overline{X} \rightarrow \mathbb{R}$ is a *viscosity* subsolution (resp. supersolution) of the *Dirichlet problem* given by (15) and (16), if it is a subsolution (resp. supersolution) of (15) in X and, any $\phi \in C^2(X)$ and any local maximum (resp. minimum) $x \in \partial X$ for $u^* - \phi$ (resp. $u_* - \phi$) $\min\{u^*(x) - k(x), F(x, u^*, D_x\phi, D_x^2\phi, \int_X u^*(y)R(x, dy)\} \leq 0$ (resp. $\max\{u_*(x)-k(x), F(x, u_*, D_x\phi, D_x^2\phi, \int_X u_*(y)R(x, dy)\} \geq 0$) where $k(x) := \int_X u(y)R(x, dy)$, $x \in \partial X$.

In general, the existence of the solutions is proved by *Perron's method*, introduced in the viscosity setting in [14]. That is, one proves that the supremum of a suitable set of subsolutions is the solution. In order to do this, one needs the help of a *comparison principle*.

B. Hamilton Jacobi Equations

Let us consider a stochastic hybrid system $H = ((Q, d, \mathcal{X}), (b, \sigma), \mu_0, (\lambda, R))$ and the reachability problem for a target set A, as defined in Section IV. For $T > 0$, we consider the reach set probability function

$$u(x, t) = P_x\{T_A \leq t, x_{T_A} \in \partial E\}, 0 \leq t \leq T$$

and its logarithmic transformation

$$h = -\ln u.$$

Proposition 2: Under the standard hypotheses that ensure the existence and uniqueness of realization M of a SHS H [6], h is the viscosity solution of the following Hamilton Jacobi equation

$$-\frac{\partial}{\partial t}h(x,t) + \mathcal{H}(x, D_x h(x,t), D_x^2 h(x,t)) = 0, \quad (x,t) \in E \times [0, T)$$

$$h(x,t) = 0, \ (x,t) \in \partial E \times [0, T]$$

$$h(x,t) \rightarrow \infty, \text{ as } t \nearrow T \text{ if } x \in E,$$

where Hamiltonian operator is given by

$$\mathcal{H}(x, D_x\varphi(x,t), D_x^2\varphi(x,t)) = = -b(x)D_x\varphi + \frac{1}{2}D_x\varphi\sigma D_x\varphi' + \int_{\partial X} \exp(\varphi(x) - \varphi(x+y))R(x, dy),$$

for any test function $\varphi \in C^2(X)$.

Obvious methods for defining the probability distributions for SHS as viscosity solutions for the Fokker Planck Kolmogorov equations can be introduced. The main advantage, in this case, is the fact that the parabolic integro-differential equations involved do not have boundary conditions that are difficult to deal with.

VII. Conclusions

In this paper, we have developed characterisations of the reach set probabilities for stochastic hybrid systems as viscosity solutions for the Dirichlet problem associated to some parabolic integro partial differential equations. The corner stone of this approach was to present the stochastic reachability problem as an exit problem for Markov processes with piecewise diffusion behaviour and forced jumps.

References

[1] Abate, A., Amin, A., Prandini, M., Lygeros, J., Sastry, S.: *Probabilistic Reachability for Discrete Time Stochastic Hybrid Systems*. Proc. IEEE CDC (2006): 258-263.

[2] Barles, C., Chasseigne, E., Imbert, C.: *On the Dirichlet Problem for Second-Order Elliptic Integro-Differential Equations*. Preprint (2007).

[3] Bect, J.: *A unifying formulation of the Fokker-Planck-Kolmogorov equation for general stochastic hybrid systems*. IFAC World Congress (2008).

[4] Blom, H.A.P., Lygeros, J. (Eds.): "*Stochastic Hybrid Systems: Theory and Safety Critical Applications*". LNCIS **337** (2006).

[5] Bujorianu, M.L., Lygeros, J.: *General Stochastic Hybrid Systems: Modelling and Optimal Control*. Proc. 43th Conference in Decision and Control (2004).

[6] Bujorianu, M.L., Lygeros, J.: *Towards Modelling of General Stochastic Hybrid Systems*. In [4]: 3-30.

[7] Bujorianu, M.L., Lygeros, J.: *New Insights on Stochastic Reachability*. Proc. 46th Conference in Decision and Control (2007).

[8] Donsker, M. D. and Varadhan, S. R. S.: *Asymptotic evaluation of certain Markov process expectations for large time*. IV. Comm. Pure. Appl. Math. (**36**) (1983).

[9] Davis, M.H.A.: "*Markov Models and Optimization*" Chapman & Hall, (1993).

[10] Ethier, S.N., Kurtz, T.G.: "*Markov Processes: Characterization and Convergence*". New York: John Wiley and Sons, (1986).

[11] Fleming, W.H., Soner, H.M.: *Asymptotic Expansions for Markov Processes with Levy Generators*. Appli. Math. Optim. (**19**) (1989): 203-223.

[12] Fleming, W.H., James, M.R.: *Asymptotic Series and Exit Time Probabilities.* Ann. Prob. (**20**) (3) (1992): 1369-1384.

[13] Fleming, W.H.: *Exit probabilities and optimal stochastic control.* Applied. Math. Optim. (**4**) (1978): 329-346.

[14] Ishii, H.: *Perron's Method for Hamilton-Jacobi Equations*. Duke Math. J., **55**(2) (1987): 369-384.

[15] Johnson, D.P.: *Diffusion approximations for optimal filtering of jump processes and for queueing networks*. Thesis (Ph. D.)–University of Wisconsin–Madison, (1983)

[16] Klosek, M.M., Matkowsky, B.J., Schuss, Z.: *First-Order Dynamics Driven by Rapid Markovian Jumps*. SIAM J. Appl. Math. **49**(6) (1989).

[17] Koutsoukos, X.; Riley, D.: *Computational Methods for Verification of Stochastic Hybrid Systems.* IEEE Trans. on Systems, Man and Cybernetics. Part A. 38(2), 385-396 (2008).

[18] Xenofon Koutsoukos, "*Optimal Control of Stochastic Hybrid Systems Based on Locally Consistent Markov Decision Processes*". 2005 IEEE International Symposium on Intelligent Control ISIC - (2005)

[19] Krystul, J., Bagchi, A.: *Approximation of first passage times of switching diffusion*. MTNS (2004).

[20] Lenhard, S.M.; Yamada, N.: *Perron's Method for Viscosity Solutions Associated with Piecewise Deterministic Processes*. Funkcialaj Ekvacioj **34** (1991): 173-186.

[21] Lygeros, J.: *On reachability and minimum cost optimal control*. Automatica **40** (6) (2004).

[22] Papanicolaou, G.C: *Asymptotic Analysis of Transport Processes*. Bull. Amer. Math. Soc. (**81**)(2) (1975): 330-292.

[23] Prandini, M., Hu, J.: "Stochastic reachability: Theoretical foundations and numerical approximation". Chapter of the book "Stochastic hybrid systems", Control Engineering Series, vol. **24**, Taylor & Francis (2006): 107-138.

[24] Yuan-jin Liu, G. Yin: *Asymptotic Expansions of Transition Densities for Hybrid Jump-diffusions*. Acta Math.Appl. Sinica (**20**)(1) (2004): 1-18.

[25] Zhu, C., Yin, G.: *Asymptotic Properties of Hybrid Diffusion Systems*. SIAM J. Control Optim., 46 (2007): 1155–1179.